

## AN EXACTLY SOLVABLE MODEL OF AN OSCILLATOR WITH NONLINEAR COUPLING AND ZEROS OF BESSEL FUNCTIONS

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### Abstract

We consider a model of oscillator with nonpolynomial interaction admitting exact solutions both for energy eigenvalues in terms of zeros of Bessel functions considered as functions of the continuous index, and for the corresponding eigenstates in terms of Lommel polynomials

Let us consider the following Hamiltonian,

$$H = \omega \alpha^\dagger \alpha + \lambda \left\{ \alpha^\dagger [\alpha^\dagger \alpha + 1]^{-1/2} + [\alpha^\dagger \alpha + 1]^{-1/2} \alpha \right\}. \quad (1)$$

Here  $\alpha$  and  $\alpha^\dagger$  are usual boson annihilation and creation operators,  $\omega$  and  $\lambda$  are positive real parameters (the generalization to complex coupling constant  $\lambda$  does not lead to any new result, since the phase of  $\lambda$  is trivially eliminated by the canonical transformation  $\alpha \rightarrow \alpha e^{i\varphi}$  preserving the energy spectrum). If the mean number of quanta is close to zero, then (1) turns into the Hamiltonian of usual forced oscillator. In the opposite quasiclassical regime of large mean number of excitations  $N = \langle \alpha^\dagger \alpha \rangle \gg 1$  the substitution  $\alpha \approx N^{1/2} e^{i\varphi}$  leads to the energy-independent interaction Hamiltonian

$$H_{\text{int}} = \lambda \cos \varphi, \quad (2)$$

which is in fact exact, since the expression inside the figure brackets is nothing but the Susskind-Glogower cosine phase operator [1] which properties were discussed in detail in the known review by Carruthers and Nieto [2].

Expanding the energy eigenstate  $|E\rangle$  over the Fock states

$$|E\rangle = \sum_{n=0}^{\infty} c_n |n\rangle \quad (3)$$

and taking into account the known matrix elements of operators  $a$  and  $a^+$  one can easily reduce the stationary Schroedinger equation to the following set of coupled linear algebraic equations,

$$\begin{aligned} Ec_0 &= \lambda c_1, \\ Ec_n &= \omega n + \lambda (c_{n-1} + c_{n+1}), \quad n \geq 1 \end{aligned} \quad (4)$$

It is convenient to introduce dimensionless variables

$$z = \lambda/\omega, \quad \mu = E/\omega. \quad (5)$$

Then normalized energy  $\mu$  is determined from the equation  $\Phi(z, \mu) = 0$ , where function  $\Phi$  is the characteristic determinant of system (4):

$$\Phi(z, \mu) = \begin{vmatrix} -\mu & z & 0 & 0 & 0 & \dots \\ z & 1-\mu & z & 0 & 0 & \dots \\ 0 & z & 2-\mu & z & 0 & \dots \\ 0 & 0 & z & 3-\mu & z & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix} \quad (6)$$

Expanding this determinant over the elements of the first row one can easily obtain the following recurrence relation,

$$\Phi(z, \mu) = -\mu \Phi(z, \mu-1) - z^2 \Phi(z, \mu-2). \quad (7)$$

Introducing new function

$$F(z, \mu) = z^{-\mu} \Phi(z, \mu) \quad (8)$$

one can rewrite (7) as follows,

$$F(z, \mu) + F(z, \mu-2) = -\frac{2\mu}{2z} F(z, \mu-1). \quad (9)$$

But this is the well known relation for Bessel functions [3,4]. Consequently, the energy levels are determined by zeros of Bessel functions in accordance with the equation

$$z^\mu J_{-1-\mu}(2z) = 0. \quad (10)$$

For small values of parameter  $z$  the well-known power series expansion of the Bessel function leads to the equation

$$z^{-1} \sum_{m=0}^{\infty} \frac{(-z^2)^m}{m! \Gamma(m-\mu)} = 0. \quad (11)$$

For  $z \rightarrow 0$  the solutions of this equation with respect to  $\mu$  are determined by the poles of gamma-function. Evidently, they reproduce

approximately the equidistant harmonic oscillator spectrum:  $\mu_n \approx n$ ;  $n = 0, 1, 2, \dots$ . Since all poles of gamma-function are simple, with the residues  $(-1)^n/n!$ , the correction to the  $n$ -th energy level has the order of  $z^{2(n+1)}$ :

$$\mu_0 \approx -z^2, \quad \mu_n = n - z^{2(n+1)}/n + \dots, \quad n \geq 1. \quad (12)$$

Note that all corrections are negative.

For large values of the coupling constant we can use the known asymptotic formula

$$J_{-1-\mu}(2z) \approx (\pi z)^{-1/2} \cos(2z + \pi\mu/2 + \pi/4). \quad (13)$$

Then for  $|\mu| \ll |z|$  the spectrum is equidistant again, but with the twice distance between the neighbouring energy levels:

$$\mu_n \approx 1/2 + 2n - 4z/\pi + O(z^{-1}). \quad (14)$$

Here  $n$  is an arbitrary integer having the same order of magnitude as the large parameter  $z$ . Note that energy values depend on the coupling constant in a specific almost periodic manner:

$$\mu_n(z) \approx \mu_{n+1}(z + \pi/2). \quad (15)$$

Now let us look again at eq. (4). Comparing it with different recurrence relations for special functions given in [3,4], one can recognize that it is nothing but the equation for Lommel's polynomials (which are in fact polynomials with respect to  $1/z$ )

$$R_{n+1,\nu}(z) + R_{n-1,\nu}(z) = \frac{2(\nu+n)}{z} R_{n,\nu}(z) \quad (16)$$

Consequently,

$$c_n^{(\mu)} = N(\mu, z) R_{n,\mu}(-2z) = N(\mu, z) \sum_{l=0}^{n/2} \frac{(-1)^{n-l} (n-l)! \Gamma(n-l-\mu)}{l! (n-2l)! \Gamma(l-\mu)} z^{2l-n}, \quad (17)$$

where  $N(\mu, z)$  is the normalizing factor. For example, the first three coefficients  $\tilde{c}_n^{(\mu)} = c_n^{(\mu)}/N(\mu, z)$  are as follows,

$$\tilde{c}_0^{(\mu)} = 1, \quad \tilde{c}_1^{(\mu)} = \mu/z, \quad \tilde{c}_2^{(\mu)} = \mu(\mu-1)/z^2 - 1 \quad (18)$$

Taking into account (12) we have, e.g., for the ground state

$$\tilde{c}_0^{(0)} = 1, \quad \tilde{c}_1^{(0)} = -2, \quad \tilde{c}_2^{(0)} = z^2, \quad (19)$$

In conclusion let us discuss the correspondence between the quantum problem under study and its classical counterpart described

in the energy - phase canonical variables

$$E = \frac{1}{2}(\rho^2 + q^2), \quad \varphi = \arccos[q/(2E)^{1/2}] \quad (20)$$

with Hamiltonian

$$H = E + \lambda \cos \varphi. \quad (21)$$

Since this Hamiltonian depends linearly on the energy variable  $E$ , the canonical equations of motion

$$\partial E / \partial t = \partial H / \partial \varphi, \quad \partial \varphi / \partial t = -\partial H / \partial E \quad (22)$$

can be found without difficulty for an arbitrary "potential"  $f(\varphi)$ :

$$\varphi(t) = -t, \quad E(t) = E_0 + f(0) - f(\varphi). \quad (23)$$

However, in the quantum case just the "potential"  $\cos \varphi$  seems distinguished. For example, if one takes instead of (2) the interaction Hamiltonian

$$\hat{H}_{\text{int}} = \lambda [\cos \hat{\varphi}]^2, \quad (24)$$

then instead of (6) and (7) one gets ( $\tilde{\mu} = \mu - 2z$ )

$$\tilde{\mathfrak{I}}(z, \mu) = \begin{vmatrix} -\tilde{\mu} & 0 & z & 0 & 0 & \dots \\ 0 & 1-\tilde{\mu} & 0 & z & 0 & \dots \\ z & 0 & 2-\tilde{\mu} & 0 & z & \dots \\ 0 & z & 0 & 3-\tilde{\mu} & 0 & z \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix}, \quad (25)$$

$$\tilde{\mathfrak{I}}(z, \mu) = -\tilde{\mu} \tilde{\mathfrak{I}}(z, \mu-1) + z^2 (\tilde{\mu}-1) \tilde{\mathfrak{I}}(z, \mu-3) + z^4 \tilde{\mathfrak{I}}(z, \mu-4) \quad (26)$$

with unknown solution.

Although the physical meaning of the quantum model with Hamiltonian (1) is not clear at the moment (its "nearest neighbour"  $H = E^2/2 + \lambda \cos \varphi$  describes the Josephson junction), we hope that due to its beauty it will find applications in future.

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